

THE SYMPLECTIC REPRESENTATION OF THE MAPPING CLASS GROUP IS UNIQUE

MUSTAFA KORKMAZ

ABSTRACT. Any nontrivial homomorphism from the mapping class group of an orientable surface of genus $g \geq 3$ to $\mathrm{GL}(2g, \mathbb{C})$ is conjugate to the standard symplectic representation. It is also shown that the mapping class group has no faithful linear representation in dimensions less than or equal to $3g - 3$.

1. INTRODUCTION AND THE RESULTS

For a compact connected orientable surface S of genus g with a finite (possibly empty) set of marked points in the interior, we define the *mapping class group* $\mathrm{Mod}(S)$ to be the group of isotopy classes of orientation-preserving self-diffeomorphisms of S . Diffeomorphisms and isotopies are assumed to be the identity on the boundary and on the marked points. Gluing a disk along each boundary component of S and forgetting the marked points give a closed surface \bar{S} and a natural surjective map $\mathrm{Mod}(S) \rightarrow \mathrm{Mod}(\bar{S})$ between the mapping class groups.

After fixing a basis for the first (integral) homology $H_1(\bar{S}, \mathbb{Z})$ of \bar{S} , the action of $\mathrm{Mod}(\bar{S})$ on $H_1(\bar{S}, \mathbb{Z})$ gives rise to a surjective homomorphism $\mathrm{Mod}(\bar{S}) \rightarrow \mathrm{Sp}(2g, \mathbb{Z})$. Precomposing this map with $\mathrm{Mod}(S) \rightarrow \mathrm{Mod}(\bar{S})$ and postcomposing with the inclusion $\mathrm{Sp}(2g, \mathbb{Z}) \hookrightarrow \mathrm{GL}(2g, \mathbb{C})$ give a map $P : \mathrm{Mod}(S) \rightarrow \mathrm{GL}(2g, \mathbb{C})$. Our first result shows that the map P is the only nontrivial representation of $\mathrm{Mod}(S)$ in this dimension:

Theorem 1. *Let $g \geq 3$ and let $\phi : \mathrm{Mod}(S) \rightarrow \mathrm{GL}(2g, \mathbb{C})$ be a group homomorphism. Then ϕ is either trivial or conjugate to the homomorphism $P : \mathrm{Mod}(S) \rightarrow \mathrm{GL}(2g, \mathbb{C})$.*

We say that two homomorphisms ψ_1 and ψ_2 from a group G to a group H are called *conjugate* if there exists an element $h \in H$ such that $\psi_2(x) = h\psi_1(x)h^{-1}$ for all $x \in G$.

One of the outstanding unsolved problems in the theory of mapping class groups is the existence of a faithful representation $\mathrm{Mod}(S) \rightarrow \mathrm{GL}(n, \mathbb{C})$. Our second result shows that in dimensions $n \leq 3g - 3$, there is no faithful linear representation of the mapping class group.

Theorem 2. *Let $g \geq 3$ and let $n \leq 3g - 3$. Then there is no injective homomorphism $\mathrm{Mod}(S) \rightarrow \mathrm{GL}(n, \mathbb{C})$.*

Date: August 17, 2011.

The linearity problem for the braid group, which is the mapping class group of a disk with marked points in the context of this paper, was solved by Bigelow [3] and also by Krammer [16, 17]. In [13], using the linearity of braid groups, the author proved that the mapping class group of the closed surface of genus 2 and of sphere with marked points are linear. These results were also obtained by Bigelow–Budney [4].

The first result on the nonexistence of faithful linear representations of the mapping class group was obtained by Farb–Lubotzky–Minsky [2] who proved that no homomorphism from a subgroup of finite index of the mapping class group into $\mathrm{GL}(n, \mathbb{C})$ is injective for $n < 2\sqrt{g-1}$. This was improved first by Funar [8] who showed that every map from the mapping class group into $\mathrm{SL}(n, \mathbb{C})$ has finite image for $n \leq \sqrt{g+1}$. This was improved further by Franks–Handel in [7] and by the author in [12], which can be rephrased as follows:

Theorem 3. [7, 12] *Let $g \geq 2$, $n \leq 2g-1$, and let $\phi : \mathrm{Mod}(S) \rightarrow \mathrm{GL}(n, \mathbb{C})$ be a homomorphism. Then ϕ factors through the natural quotient map $\mathrm{Mod}(S) \rightarrow H_1(\mathrm{Mod}(S), \mathbb{Z})$. In particular, $\mathrm{Im}(\phi)$ is trivial if $g \geq 3$, and is a quotient of the cyclic group \mathbb{Z}_{10} of order 10 if $g = 2$.*

We state a few algebraic corollaries to above theorems, some of which might be known to the experts.

Corollary 4. *If $g \geq 3$ then up to the conjugation by an element of $\mathrm{GL}(2g, \mathbb{C})$ there are exactly two homomorphisms $\mathrm{Sp}(2g, \mathbb{Z}) \rightarrow \mathrm{GL}(2g, \mathbb{C})$, the trivial homomorphism and the injection given by the inclusion.*

Corollary 5. *If $g \geq 3$ and $n \leq 2g$ then every homomorphism from $\mathrm{PSp}(2g, \mathbb{Z})$ to $\mathrm{GL}(n, \mathbb{C})$ is trivial.*

Corollary 6. *If $g \geq 3$ and if Q is a finite quotient of $\mathrm{Mod}(S)$, then every homomorphism $Q \rightarrow \mathrm{GL}(2g, \mathbb{C})$ is trivial.*

In order to prove these corollaries, consider the composition the given homomorphism $H \rightarrow \mathrm{GL}(2g, \mathbb{C})$ with the natural surjective map $\mathrm{Mod}(S) \rightarrow H$ and apply Theorem 1 or Theorem 3.

Here is an outline of the paper. In Section 2, we give some lemmas on matrices. Section 3 gives the relevant background from the theory of mapping class groups and surface topology. Section 4 investigates the properties of eigenvalues and eigenspaces of the image of a Dehn twist about a nonseparating simple closed curve. The main result in this section is Lemma 4.7, which constitutes a major step in the proofs of both theorems. The first main result, Theorem 1, is proved in Section 5. We give the proof for $g \geq 4$ first, and an then outline for $g = 3$. Finally, Theorem 2 is proved in Section 6.

Acknowledgments. This paper was written while I was visiting Max-Planck Institut für Mathematik in Bonn on leave from Middle East Technical University. I thank MPIM for its generous support and wonderful research

environment. I also thank Çağrı Karakurt for the discussions and for his comments.

2. PRELIMINARIES IN LINEAR ALGEBRA

In this section, we give the necessary definitions, setup the notation and prove some useful lemmas on matrices. We denote by \bar{S} the closed surface obtained from S by gluing a disk along each boundary component.

Throughout the paper, the letters U and \widehat{U} will always denote the 2×2 matrices $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\widehat{U} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

Definition 2.1. For each $i = 1, 2, \dots, g$, we define the matrices A_i to be the $2g \times 2g$ block diagonal matrix $\text{Diag}(I_2, \dots, I_2, U, I_2, \dots, I_2)$ and B_i to be the $2g \times 2g$ block diagonal matrix $\text{Diag}(I_2, \dots, I_2, \widehat{U}, I_2, \dots, I_2)$. Here, U and \widehat{U} are in the i^{th} block on the diagonal, and I_2 is the 2×2 identity matrix.

We note that A_i (resp. B_i) is the symplectic matrix of the action of the Dehn twist t_{a_i} (resp. t_{b_i}) on the first homology group of \bar{S} with respect to the basis $\{a_1, b_1, a_2, b_2, \dots, a_g, b_g\}$ of $H_1(\bar{S}, \mathbb{Z})$, where a_i and b_i are the (oriented) simple closed curves given in Figure 2. Recall that when we consider a Dehn twist about a curve, the orientation of the curve is unimportant.

Lemma 2.2. *Let X , Y and Z be, respectively, $2 \times k$, $k \times 2$ and 2×2 , matrices with entries in \mathbb{C} .*

- (i) *If $UX = \widehat{U}X = X$, then $X = 0$.*
- (ii) *If $YU = Y\widehat{U} = Y$, then $Y = 0$.*
- (iii) *If $ZU = UZ$ and $Z\widehat{U} = \widehat{U}Z$, then $Z = aI$.*

Proof. The proof is straight forward. □

Lemma 2.3. *Let X , Y and Z be matrices with entries in \mathbb{C} such that the given multiplication are defined:*

- (i) *If $A_iX = B_iX = X$ for all i , then $X = 0$.*
- (ii) *If $YA_i = YB_i = Y$ for all i , then $Y = 0$.*
- (iii) *If $ZA_i = A_iZ$ and $ZB_i = B_iZ$ for all i , then Z is equal to a diagonal matrix $\text{Diag}(a_1I_2, a_2I_2, \dots, a_gI_2)$ for some $a_i \in \mathbb{C}$.*

Proof. This lemma is a slight generalization of Lemma 2.2, and may be proved easily by induction on g . □

Lemma 2.4. *Let $X \in \text{GL}(2, \mathbb{C})$. Suppose that $XU = UX$ and $X\widehat{U}X = \widehat{U}X\widehat{U}$. Then $X = U$.*

Proof. Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The equality $XU = UX$ implies that $c = 0$ and $d = a$, so that $X = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. Note that $a \neq 0$. Now the equality $X\widehat{U}X = \widehat{U}X\widehat{U}$ gives the equations

$$\begin{aligned} a(a-b) &= a-b, \\ a^2 &= 2a-b, \\ b(2a-b) &= b. \end{aligned}$$

The only solution of these equations is $a = b = 1$. \square

Lemma 2.5. *Let $X \in \text{GL}(2g, \mathbb{C})$. Suppose that*

- (i) *all eigenvalues of X are equal to 1,*
- (ii) *$XA_i = A_iX$ for all $i = 1, 2, \dots, g$,*
- (iii) *$XB_j = B_jX$ for all $j = 2, 3, \dots, g$, and*
- (iv) *$XB_1X = B_1XB_1$.*

Then $X = A_1$.

Proof. If $g = 1$ then the lemma reduces to Lemma 2.4. So suppose that $g \geq 2$.

Let us write $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$, where X_1 and X_4 are, respectively, 2×2 and $(2g-2) \times (2g-2)$ matrices. For each $i \geq 2$, we define \bar{A}_i and \bar{B}_i by $A_i = \begin{pmatrix} I_2 & 0 \\ 0 & \bar{A}_i \end{pmatrix}$, $B_i = \begin{pmatrix} I_2 & 0 \\ 0 & \bar{B}_i \end{pmatrix}$.

It is easy to see that, for $i \geq 2$, the equations $XA_i = A_iX$ and $XB_i = B_iX$ imply that

- $X_2\bar{A}_i = X_2\bar{B}_i = X_2$,
- $\bar{A}_iX_3 = \bar{B}_iX_3 = X_3$, and
- $X_4\bar{A}_i = \bar{A}_iX_4$ and $X_4\bar{B}_i = \bar{B}_iX_4$.

It now follows from Lemma 2.3 and the assumption that all eigenvalues of X are equal to 1 that $X_2 = 0$, $X_3 = 0$ and $X_4 = I_{2g-2}$.

Moreover, from the equations $XA_1 = A_1X$ and $XB_1X = B_1XB_1$, we obtain

- $X_1U = UX_1$, and
- $X_1\widehat{U}X_1 = \widehat{U}X_1\widehat{U}$.

Now these two equations and Lemma 2.4 give us the equality $X_1 = U$, so that $X = A_1$, which is the desired result. \square

Remark 2.6. The above proof may be modified easily to prove the following version of Lemma 2.5: Suppose that

- (i) *all eigenvalues of $X \in \text{GL}(2g, \mathbb{C})$ are equal to 1,*
- (ii) *$XA_i = A_iX$ for all $i = 1, 2, \dots, g$,*
- (iii) *$XB_j = B_jX$ for all $1 \leq j \leq g$ with $j \neq k$, and*
- (iv) *$XB_kX = B_kXB_k$.*

Then $X = A_k$. The lemma holds true when the roles of A_i and B_i are exchanged as well.

We state the following facts which will be used throughout the paper.

Lemma 2.7. *Let r and s be two positive integers. Then the subgroup of $\mathrm{GL}(r+s, \mathbb{C})$ consisting of the matrices of the form $\begin{pmatrix} I_r & * \\ 0 & I_s \end{pmatrix}$ is abelian.*

Lemma 2.8. *The subgroup of $\mathrm{GL}(m, \mathbb{C})$ consisting of upper triangular matrices is solvable.*

3. THE MAPPING CLASS GROUP RESULTS

Let S be a compact connected oriented surface of genus g with a finite number of marked points in the interior. In this section we state the results from the theory of mapping class groups that are needed in the proofs of our main results. For further information on mapping class groups, the reader is referred to [10], or [1]. For a simple closed curve a on S we denote by t_a the (isotopy class of the) right Dehn twist about a .

Theorem 3.1. ([15], Theorem 1.2) *Let $g \geq 1$ and let a and b be two nonseparating simple closed curves on S . Then there is a sequence*

$$a = a_0, a_1, a_2, \dots, a_k = b$$

of nonseparating simple closed curves such that a_{i-1} intersects a_i at only one point

It is known that for $g \geq 2$, the group $\mathrm{Mod}(S)$ is generated Dehn twists about nonseparating simple closed curves. If the surface is closed this is due to Dehn [6] and Lickorish [18]. We record this fact and three well-known relations among Dehn twists.

Theorem 3.2. *If $g \geq 2$ then the mapping class group $\mathrm{Mod}(S)$ is generated Dehn twists about nonseparating simple closed curves on S .*

We note that the above theorem does not hold true for $g = 1$. More precisely, if S is a torus with at least two boundary components, then Dehn twists about nonseparating simple closed curves are not sufficient to generate $\mathrm{Mod}(S)$: one also needs Dehn twists about the curves parallel to the boundary components [9, 14].

Lemma 3.3. *Let a and b be two simple closed curves on S , and let t_a and t_b denote the right Dehn twists about them.*

- (1) *If a and b are disjoint, then t_a and t_b commute.*
- (2) (**Braid relation**) *If a intersects b transversely at one point, then they satisfy the braid relation $t_a t_b t_a = t_b t_a t_b$.*

- (3) (**Lantern relation**) Consider a sphere X with four boundary components a, b, c, d . Let x, y, z be three simple closed curves on X as shown in Figure 1. Then the Dehn twists about them satisfy the lantern relation

$$t_a t_b t_c t_d = t_x t_y t_z.$$

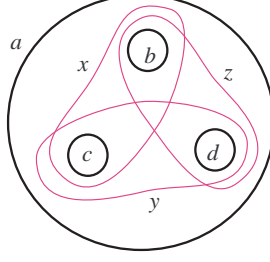


FIGURE 1. The curves of the lantern relation.

We remind that the first homology group $H_1(G; \mathbb{Z})$ of a group G is isomorphic to the abelianization $G/[G, G]$, where $[G, G]$ is the (normal) subgroup of G generated by all commutators $xyx^{-1}y^{-1}$, where $x, y \in G$.

Theorem 3.4. ([14], Theorem 5.1) *If $g \geq 2$ then first homology group $H_1(\text{Mod}(S); \mathbb{Z})$ is*

- (1) *trivial if $g \geq 3$, and*
- (2) *isomorphic to the cyclic group of order 10 if $g = 2$.*

If $g \geq 3$ then the group $\text{Mod}(S)$ is perfect. In the case $g = 2$, $\text{Mod}(S)$ is not perfect but its commutator subgroup is perfect.

Theorem 3.5. ([15], Theorem 4.2) *If $g \geq 2$ then the commutator subgroup of $\text{Mod}(S)$ is perfect.*

Theorem 3.6. ([15], Theorem 2.7) *Let $g \geq 2$ and let a and b be two nonseparating simple closed curves on S intersecting at one point. Then the commutator subgroup of $\text{Mod}(S)$ is generated normally by $t_a t_b^{-1}$.*

Note that since any two Dehn twists about nonseparating simple closed curves are conjugate in $\text{Mod}(S)$, they represent the same element in the group $H_1(\text{Mod}(S); \mathbb{Z})$. In particular, we conclude the next lemma.

Lemma 3.7. *Let $g \geq 1$, and let b and c be two nonseparating simple closed curves on S . If H is an abelian group and if $\phi : \text{Mod}(S) \rightarrow H$ is a homomorphism, then $\phi(t_b) = \phi(t_c)$.*

Let us write $S = S_{g,r}^p$, for the moment, for the surface of genus $g \geq 2$ with $p \geq 0$ boundary and with $r \geq 0$ marked points in the interior. For $r \geq 1$ by forgetting one of the marked points, we get a short exact sequence,

$$(1) \quad 1 \longrightarrow \pi_1(S_{g,r-1}^p) \longrightarrow \text{Mod}(S_{g,r}^p) \longrightarrow \text{Mod}(S_{g,r-1}^p) \longrightarrow 1,$$

where the map from the fundamental group is obtained by pushing the base point along the given path.

For $p \geq 1$, by blowing down a given boundary component d to a marked point, we get a short exact sequence

$$(2) \quad 1 \longrightarrow \mathbb{Z} \longrightarrow \text{Mod}(S_{g,r}^p) \longrightarrow \text{Mod}(S_{g,r+1}^{p-1}) \longrightarrow 1,$$

where the group \mathbb{Z} is generated by the Dehn twist t_d about the boundary component d . These two exact sequences are called *Birman's exact sequences*.

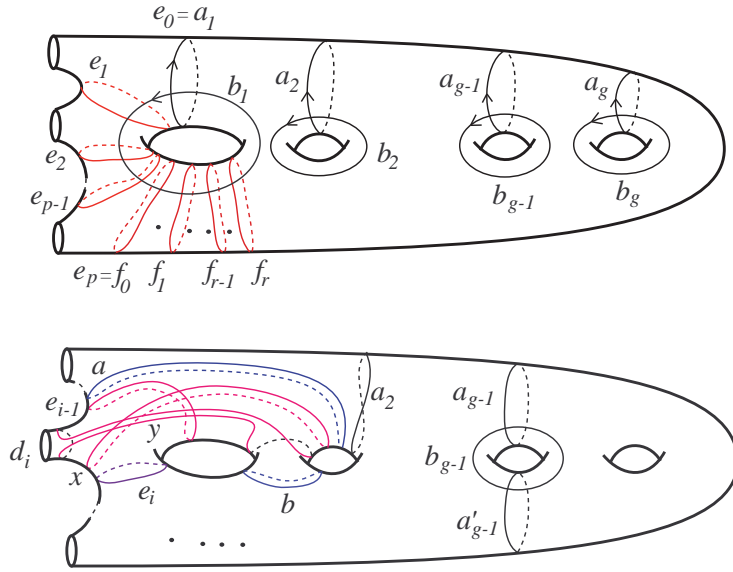


FIGURE 2. Various curves on the surface and a lantern.

Let $g \geq 2$, and let $\bar{S} = S_{g,0}^0$ be the closed surface obtained from S by gluing a disk along each boundary component. Let $\pi : \text{Mod}(S) \rightarrow \text{Mod}(\bar{S})$ be the natural surjective homomorphism obtained by extending diffeomorphism $S \rightarrow S$ to $\bar{S} \rightarrow \bar{S}$ by the identity on the disks glued. Let d_1, d_2, \dots, d_p be (the simple closed curves parallel to) the boundary components of S . Consider the simple closed curves $e_0, e_1, \dots, e_p, f_0, f_1, \dots, f_r$ and a'_{g-1} illustrated in Figure 2. Hence, the union $d_i \cup e_{i-1} \cup e_i$ bounds a pair of pants, $f_{j-1} \cup f_j$ bounds an annulus with a marked point, and $a_{g-1} \cup a'_{g-1}$ bounds a subsurface diffeomorphic to a torus with two boundary components. Clearly, for $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, r$, the mapping classes $t_{d_i}, t_{e_{i-1}} t_{e_i}^{-1}$ and $t_{f_{j-1}} t_{f_j}^{-1}$ are contained in the kernel of π . It can be shown by using Birman's exact sequences that the kernel of π is generated normally by

$$\{t_{d_i}, t_{e_{i-1}} t_{e_i}^{-1}, t_{f_{j-1}} t_{f_j}^{-1} : 1 \leq i \leq p, 1 \leq j \leq r\}.$$

We state this fact as a proposition which will be useful for us. We also show that we may omit the Dehn twists about the boundary parallel curves.

Proposition 3.8. *The kernel of the natural surjective homomorphism $\pi : \text{Mod}(S) \rightarrow \text{Mod}(\bar{S})$ is generated normally by the set*

$$(3) \quad \{t_{e_{i-1}}t_{e_i}^{-1}, t_{f_{j-1}}t_{f_j}^{-1} : 1 \leq i \leq p, 1 \leq j \leq r\}.$$

Proof. For the proof, it suffices to prove that each t_{d_i} can be written as a product of conjugates of elements in (3). We use the lantern relation for this. Consider the curves in the bottom picture of Figure 2. The curves d_i, e_i, a, b bound a sphere with four boundary components. The curves on this sphere satisfy the lantern relation

$$t_{d_i}t_{e_i}t_at_b = t_{e_{i-1}}t_xt_y,$$

which may be rewritten as

$$t_{d_i} = (t_{e_{i-1}}t_{e_i}^{-1})(t_xt_a^{-1})(t_yt_b^{-1}).$$

Since the pair (x, a) can be mapped (e_{i-1}, e_i) by a diffeomorphism of S , it follows that the element $t_xt_a^{-1}$ is conjugate to $t_{e_{i-1}}t_{e_i}^{-1}$. By the similar reasoning, $t_yt_b^{-1}$ is also conjugate to $t_{e_{i-1}}t_{e_i}^{-1}$. \square

4. EIGENVALUES AND EIGENSPACES OF $\phi(t_a)$

Let $\phi : \text{Mod}(S) \rightarrow \text{GL}(m, \mathbb{C})$ be a homomorphism. For a simple closed curve a on S , we denote by L_a the image $\phi(t_a)$ of the Dehn twist t_a . If λ is an eigenvalue of L_a , the eigenspace corresponding to λ is denoted by E_λ^a .

For each $i = 1, 2, \dots, g$, let a_i and b_i be the nonseparating simple closed curves on S shown in Figure 2. We may consider them on \bar{S} as well, where \bar{S} is the closed surface obtained from S by gluing a disk along each boundary component. The homology classes of the orient curves a_i and b_i form a basis for the first homology group $H_1(\bar{S}, \mathbb{Z})$ of \bar{S} . In order to avoid double subscript, we write $L_{2i-1} = \phi(t_{a_i})$ and $L_{2i} = \phi(t_{b_i})$.

For an eigenvalue λ of a matrix M , let $\lambda_\#(M)$ denote the multiplicity of λ in the characteristic polynomial of M . We will omit M from the notation and write $\lambda_\#$ only; the matrix M will always be clear from the context.

Lemma 4.1. *Let L and M be two linear automorphisms of \mathbb{C}^m with $LM = ML$. If λ is an eigenvalue of L , then $\ker(L - \lambda I)^k$ is M -invariant for all $k \geq 1$. In particular, $E_\lambda = \ker(L - \lambda I)$ is M -invariant.*

Lemma 4.2. ([12]) *Let $\phi : \text{Mod}(S) \rightarrow \text{GL}(m, \mathbb{C})$ be a homomorphism. Let a, b, c, d be nonseparating simple closed curves on S such that there exists an orientation-preserving diffeomorphism $f : S \rightarrow S$ with $f(c) = a$ and $f(d) = b$. Suppose that λ is an eigenvalue of $L_a = \phi(t_a)$. Then $E_\lambda^a = E_\lambda^b$ if and only if $E_\lambda^c = E_\lambda^d$.*

Lemma 4.3. *Let S be a compact connected oriented surface of genus $g \geq 1$ and let $\phi : \text{Mod}(S) \rightarrow \text{GL}(m, \mathbb{C})$ be a homomorphism. Suppose that there are two nonseparating simple closed curves a and b on S intersecting at one point such that $E_\lambda^a = E_\lambda^b$ for some eigenvalue λ of L_a and L_b . Then E_λ^a is $\text{Mod}(S)$ -invariant. That is, $\phi(f)(E_\lambda^a) = E_\lambda^a$ for all $f \in \text{Mod}(S)$.*

Proof. Let x be a nonseparating simple closed curve on S . By Theorem 3.1, there is a sequence $a = a_0, a_1, a_2, \dots, a_k = x$ of nonseparating simple closed curves such that a_{i-1} intersects a_i at one point for all $1 \leq i \leq k$. Since, by the classification of surfaces, there exists a diffeomorphism f_i of S mapping (a, b) to (a_{i-1}, a_i) , we have $E_\lambda^{a_{i-1}} = E_\lambda^{a_i}$ by Lemma 4.2. It follows that $E_\lambda^x = E_\lambda^a$ for all nonseparating simple closed curves x . Since $\text{Mod}(S)$ is generated by nonseparating Dehn twists, we conclude that the subspace E_λ^a is $\text{Mod}(S)$ -invariant. \square

Lemma 4.4. *Let $g \geq 3$, let $\phi : \text{Mod}(S) \rightarrow \text{GL}(m, \mathbb{C})$ be a homomorphism and let a be a nonseparating simple closed curve on S . Suppose that L_a has only one eigenvalue λ with $\dim(E_\lambda^a) = m - 1$. Then $\lambda = 1$.*

Proof. Choose six nonseparating simple closed curves $c_1, c_2, c_3, c_4, c_5, c_6$ on S disjoint from a such that we have the following lantern relation:

$$t_a t_{c_1} t_{c_2} t_{c_3} = t_{c_4} t_{c_5} t_{c_6}.$$

Let $\alpha = \{v_1, v_2, \dots, v_m\}$ be a basis of \mathbb{C}^m with $v_j \in E_\lambda^a$ for $j \geq 2$, so that

$$L_a = \begin{pmatrix} \lambda & 0 \\ \star & \lambda I_{m-1} \end{pmatrix}$$

with respect to α . Since each Dehn twist t_{c_i} is conjugate to t_a , $L_{c_i} = \phi(t_{c_i})$ is conjugate to L_a and, hence, it has unique eigenvalue λ . Since L_{c_i} commutes with L_a , it preserves the eigenspace E_λ^a ; $L_{c_i}(E_\lambda^a) = E_\lambda^a$. Thus, the matrix of L_{c_i} with respect to the basis α is

$$L_{c_i} = \begin{pmatrix} \lambda & 0 \\ \star & C_i \end{pmatrix},$$

where C_i is a square matrix of size $m - 1$. Now the lantern relation

$$L_a L_{c_1} L_{c_2} L_{c_3} = L_{c_4} L_{c_5} L_{c_6}$$

implies that $\lambda^4 = \lambda^3$, giving $\lambda = 1$. \square

Lemma 4.5. *Let $g \geq 3$, let $\phi : \text{Mod}(S) \rightarrow \text{GL}(m, \mathbb{C})$ be a homomorphism and let a be a nonseparating simple closed curve on S . If μ is an eigenvalue of $L_a = \phi(t_a)$ with $\mu_\# \leq 2g - 3$ then $\mu = 1$ and the dimension of the eigenspace E_μ^a is $\mu_\#$.*

Proof. If $m \leq 2g - 1$ then the image of ϕ is trivial by Theorem 3. In particular, all eigenvalues of L_a are equal to 1. Now assume that $m \geq 2g$.

Let $\mu = \mu_0, \mu_1, \mu_2, \dots, \mu_s$ be all distinct eigenvalues of L_a . Set

$$K = \ker(L_a - \mu_0 I)^m, \text{ and } K' = \bigoplus_{i=1}^s \ker(L_a - \mu_i I)^m$$

so that $\dim(K) = \mu_{\#}$, and that $\mathbb{C}^m = K \oplus K'$. Let α be a basis of K and α' be a basis of K' such that with respect to the basis $\alpha \cup \alpha'$ of \mathbb{C}^m , the matrix of L_a

$$L_a = \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix},$$

is in Jordan form, where A is a square matrix of size $\mu_{\#}$.

Let R be the complement of a regular neighborhood of a . Extending self-diffeomorphisms of R to S by the identity gives a homomorphism $q : \text{Mod}(R) \rightarrow \text{Mod}(S)$. Let $\psi = \phi q$ and let a' be a simple closed curve on R which is isotopic to a on S . If $f \in \text{Mod}(R)$ is any element, $\psi(f)$ commutes with L_a so that it preserves the subspaces K and K' . Hence, the matrix of $\psi(f)$ is

$$\psi(f) = \begin{pmatrix} F & 0 \\ 0 & F' \end{pmatrix},$$

where F is a square matrix of size $\mu_{\#}$.

Now, the correspondence $f \mapsto F$ defines a homomorphism $\bar{\psi} : \text{Mod}(R) \rightarrow \text{GL}(K) = \text{GL}(\mu_{\#}, \mathbb{C})$. Since $\mu_{\#} \leq 2g - 3 = 2(g - 1) - 1$ and the genus of R is $g - 1 \geq 2$, the image of the map $\bar{\psi}$ is cyclic by Theorem 3. It is easy to find six simple closed curves b, c, d, x, y, z which are nonseparating on R such that there is the lantern relation $t_a t_b t_c t_d = t_x t_y t_z$, and that each of $t_x t_b^{-1}$, $t_y t_c^{-1}$ and $t_z t_d^{-1}$ is a commutator in $\text{Mod}(R)$. Since $t_{a'} = t_x t_b^{-1} t_y t_c^{-1} t_z t_d^{-1}$, it follows that $t_{a'}$ is contained in the commutator subgroup of $\text{Mod}(R)$. In particular, $\bar{\psi}(t_{a'}) = I$. As a result of this we have

$$L_a = L_{a'} = \begin{pmatrix} \bar{\psi}(t_{a'}) & 0 \\ 0 & A' \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & A' \end{pmatrix}.$$

Hence, $\mu = 1$ and $\dim(E_{\mu}^a) = \mu_{\#}$. □

Corollary 4.6. *Let g, ϕ and a be as in Lemma 4.5. If $m \leq 4g - 5$ then L_a has at most two eigenvalues.*

4.1. The main lemma. The main step in the proof of Theorem 1 is the next lemma, which is also used in the proof of Theorem 2.

Lemma 4.7. *Let $g \geq 1$, $m \geq 2g$ and let $\phi : \text{Mod}(S) \rightarrow \text{GL}(m, \mathbb{C})$ be a homomorphism. Let a be a nonseparating simple closed curve on S . Suppose that the Jordan form of L_a is*

$$(4) \quad \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & \\ \hline 0 & & I_{m-2} \end{array} \right).$$

Suppose also that there exists a nonseparating simple closed curve b intersecting a at one point such that $E_1^a \neq E_1^b$. Then there is a basis of \mathbb{C}^m with respect to which

$$L_{a_i} = \begin{pmatrix} A_i & 0 \\ 0 & I \end{pmatrix} \text{ and } L_{b_i} = \begin{pmatrix} B_i & 0 \\ 0 & I \end{pmatrix},$$

where I is the identity matrix of size $m-2g$. (See Definition 2.1 for A_i and B_i , and Figure 2 for the curves a_i and b_i .)

Proof. For $i = 1, 2, \dots, g$, we set

$$L_{2i-1} = L_{a_i}, L_{2i} = L_{b_i}, E^{2i-1} = E_1^{a_i} \text{ and } E^{2i} = E_1^{b_i}.$$

We also set

$$\tilde{A}_i = \begin{pmatrix} A_i & 0 \\ 0 & I \end{pmatrix} \text{ and } \tilde{B}_i = \begin{pmatrix} B_i & 0 \\ 0 & I \end{pmatrix}.$$

Since $E_1^a \neq E_1^b$ and since there exists a homeomorphism mapping (a, b) to (a_i, b_i) , by Lemma 4.2 we have $E^{2i-1} \neq E^{2i}$ for all $1 \leq i \leq g$. Note that $\dim(E^j) = m-1$ for all $1 \leq j \leq 2g$.

Since $E^1 \neq E^2$, the dimension of $W_1 = E^1 \cap E^2$ is $m-2$. Let $\{v_3, v_4, \dots, v_m\}$ be a basis of W_1 . Choose $v_1 \in E^1 \setminus E^2$ and $v_2 \in E^2 \setminus E^1$, so that $\beta_0 = \{v_1, v_2, \dots, v_m\}$ is an ordered basis of \mathbb{C}^m . With respect to β_0 , we have

$$L_1 = \left(\begin{array}{cc|c} 1 & x_1 & 0 \\ 0 & 1 & \\ \hline 0 & X & I_{m-2} \end{array} \right), \quad L_2 = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ y'_2 & 1 & \\ \hline Y' & 0 & I_{m-2} \end{array} \right),$$

where $X = (x_3 \ x_4 \ \dots \ x_m)^t$.

If $x_1 = 0$ then we conclude from the braid relation

$$(5) \quad L_1 L_2 L_1 = L_2 L_1 L_2$$

that $y'_2 = 0$. But then L_1 and L_2 commute. Again from the relation (5) we get $L_1 = L_2$, contradicting to $E^1 \neq E^2$. Hence, x_1 is nonzero.

Let $w_1 = x_1 v_1 + x_3 v_3 + x_4 v_4 + \dots + x_m v_m$. Let β'_0 be the basis obtained from β_0 by replacing v_1 with w_1 . With respect to this new basis we have

$$L_1 = \begin{pmatrix} U & 0 \\ 0 & I_{m-2} \end{pmatrix} = \tilde{A}_1, \quad L_2 = \begin{pmatrix} 1 & 0 & 0 \\ y_2 & 1 & \\ \hline Y & 0 & I_{m-2} \end{pmatrix},$$

where $Y = (y_3 \ y_4 \ \cdots \ y_m)^t$.

From the braid relation (5), it is easy to conclude that $y_2 = -1$. Now let $w_2 = v_2 - (y_3v_3 + y_4v_4 + \cdots + y_mv_m)$ and let β_1 be the basis β'_0 where v_2 is replaced with w_2 . With respect to the basis $\beta_1 = \{w_1, w_2, v_3, v_4, \dots, v_{n-1}, v_m\}$ we now have $L_1 = \tilde{A}_1$ and $L_2 = \tilde{B}_1$.

Suppose that $k < g$ and that there is a basis

$$\beta_k = \{v_1, v_2, \dots, v_{m-1}, v_m\}$$

with respect to which

$$(6) \quad L_{2i-1} = \tilde{A}_i \text{ and } L_{2i} = \tilde{B}_i$$

for all $i = 1, 2, \dots, k$. Note that in this case

$$\alpha = \{v_{2k+1}, v_{2k+2}, \dots, v_{m-1}, v_m\}$$

is contained in $W_k = \bigcap_{i=1}^{2k} E^i$. It can be shown easily that, in fact, α is a basis for W_k , so that $\dim(W_k) = m - 2k$.

Next, we consider L_{2k+1} and L_{2k+2} . Let $s \in \{2k+1, 2k+2\}$. Since the subspace W_k is L_s -invariant, with respect to the basis β_k ,

$$L_s = \begin{pmatrix} Z_s & 0 \\ Y_s & X_s \end{pmatrix}.$$

Here, Z_s is a $2k \times 2k$ matrix. Since L_s commutes with each

$$L_{2i-1} = \begin{pmatrix} \bar{A}_i & 0 \\ 0 & I \end{pmatrix} \text{ and } L_{2i} = \begin{pmatrix} \bar{B}_i & 0 \\ 0 & I \end{pmatrix}$$

for $i = 1, 2, \dots, k$, where the matrix \bar{A}_i is the $2k \times 2k$ block diagonal matrix $\text{Diag}(I_2, \dots, U, \dots, I_2)$ whose i^{th} block is U , and \bar{B}_i is obtained from \bar{A}_i by replacing U with \hat{U} , we get that

- $Z_s \bar{A}_i = \bar{A}_i Z_s, Y_s \bar{A}_i = Y_s,$
- $Z_s \bar{B}_i = \bar{B}_i Z_s, Y_s \bar{B}_i = Y_s$

for each i . We conclude from Lemma 2.3 that $Z_s = I_{2k}$ and $Y_s = 0$, so that

$$L_s = \begin{pmatrix} I_{2k} & 0 \\ 0 & X_s \end{pmatrix}$$

In particular, v_1, v_2, \dots, v_{2k} are eigenvectors of L_s .

If W_k were a subspace of E^s , then we would have $\dim(E^s) = m$. By this contradiction, both of the subspaces $W_k \cap E^{2k+1}$ and $W_k \cap E^{2k+2}$ are of dimension $m - 2k - 1$. If, furthermore, we had $W_k \cap E^{2k+1} = W_k \cap E^{2k+2}$, then we would conclude that $E^{2k+1} = E^{2k+2}$, again arriving at a contradiction. Hence, the subspaces $W_k \cap E^{2k+1}$ and $W_k \cap E^{2k+2}$ are different, so that

$$W_{k+1} = W_k \cap E^{2k+1} \cap E^{2k+2}$$

is of dimension $m - 2k - 2$.

Let $\{w_{2k+3}, w_{2k+4}, \dots, w_m\}$ be a basis of W_{k+1} . We choose two vectors w_{2k+1} and w_{2k+2} such that

- $w_{2k+1} \in W_k \cap E^{2k+1}$, $w_{2k+1} \notin W_{2k+1}$,
- $w_{2k+2} \in W_k \cap E^{2k+2}$, $w_{2k+2} \notin W_{2k+1}$.

Then $\{w_{2k+1}, w_{2k+2}, w_{2k+3}, w_{2k+4}, \dots, w_{m-1}, w_m\}$ is a basis of W_k . Now consider the basis

$$\bar{\beta}_k = \{v_1, v_2, \dots, v_{2k}, w_{2k+1}, w_{2k+2}, \dots, w_{m-1}, w_m\}$$

of \mathbb{C}^m . With respect to this basis,

- $L_{2i-1} = \tilde{A}_i$, $L_{2i} = \tilde{B}_i$ for $i = 1, 2, \dots, k$,
- $L_{2k+1} = \begin{pmatrix} I_{2k} & 0 & 0 \\ 0 & X_1 & 0 \\ 0 & X_2 & I \end{pmatrix}$ and $L_{2k+2} = \begin{pmatrix} I_{2k} & 0 & 0 \\ 0 & Y_1 & 0 \\ 0 & Y_2 & I \end{pmatrix}$,

where $X_1 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ x_{2k+3} & x_{2k+4} & \cdots & x_m \end{pmatrix}^t$, $Y_1 = \begin{pmatrix} 1 & 0 \\ y' & 1 \end{pmatrix}$, and $Y_2 = \begin{pmatrix} y'_{2k+3} & y'_{2k+4} & \cdots & y'_m \\ 0 & 0 & \cdots & 0 \end{pmatrix}^t$.

The rest of the proof proceeds as above: If $x = 0$ then the braid relation

$$(7) \quad L_{2k+1}L_{2k+2}L_{2k+1} = L_{2k+2}L_{2k+1}L_{2k+2}$$

would imply that $y' = 0$, so that L_{2k+1} and L_{2k+2} commute. The braid relation (7) now gives $L_{2k+1} = L_{2k+2}$, which is a contradiction.

Hence, $x \neq 0$. Define

$$w'_{2k+1} = x w_{2k+1} + (x_{2k+3}w_{2k+3} + x_{2k+4}w_{2k+4} + \cdots + x_m w_m)$$

and let β'_k be the basis obtained from $\bar{\beta}_k$ by replacing w_{2k+1} with w'_{2k+1} . With respect to β'_k , the matrices of L_1, \dots, L_{2k} are the same, and $L_{2k+1} = \tilde{A}_{k+1}$. The matrix of L_{2k+2} turns into a new matrix of the form above where y' is replaced by some y , and y'_j is replaced by some y_j . The braid relation (7) then implies that $y = -1$. If we now define

$$w'_{2k+2} = w_{2k+2} - (y_{2k+3}w_{2k+3} + y_{2k+4}w_{2k+4} + \cdots + y_m w_m)$$

and

$$\beta_{k+1} = \{v_1, v_2, \dots, v_{2k}, w'_{2k+1}, w'_{2k+2}, w_{2k+3}, w_{2k+4}, \dots, w_{m-1}, w_m\},$$

we have $L_{2i-1} = \tilde{A}_i$ and $L_{2i} = \tilde{B}_i$ for all $1 \leq i \leq k+1$ with respect to β_{k+1} .

Consequently, repeating this for $k = 1, 2, 3, \dots, g-1$, with respect to some basis of \mathbb{C}^m , $L_{2i-1} = \tilde{A}_i$ and $L_{2i} = \tilde{B}_i$ for all $1 \leq i \leq g$.

This finishes the proof of the lemma. \square

4.2. Triviality of a representation of mapping class group. We give some criteria for the triviality of a representation of the mapping class group into $\mathrm{GL}(m, \mathbb{C})$. The main tool for this is Theorem 3. This subsection is inspired by [7].

Lemma 4.8. *Let S be a compact connected oriented surface of genus $h \geq 2$ and let $\psi : \mathrm{Mod}(S) \rightarrow \mathrm{GL}(m, \mathbb{C})$ be a homomorphism. If there is a flag $0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_k = \mathbb{C}^m$ of $\mathrm{Mod}(S)$ -invariant subspaces such that $\dim(W_i/W_{i-1}) \leq 2h - 1$ for each $i = 1, 2, \dots, k$, then the image of ψ is*

- (i) *trivial if $h \geq 3$, and*
- (ii) *a quotient of \mathbb{Z}_{10} if $h = 2$.*

Equivalently, the image of the commutator subgroup of $\mathrm{Mod}(S)$ is trivial.

Proof. We set $\Gamma = \mathrm{Mod}(S)$. For each $i = 1, 2, \dots, k$, let α_i be a basis of W_i such that $\alpha_i \subset \alpha_{i+1}$. We work with the basis α_k of $W_k = \mathbb{C}^m$. Since each W_i is Γ -invariant, for $f \in \Gamma$ the matrix $\phi(f)$ is of the form

$$\psi(f) = \begin{pmatrix} F_1 & \star & \star & \cdots & \star \\ 0 & F_2 & \star & \cdots & \star \\ 0 & 0 & F_3 & \cdots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & F_k \end{pmatrix}.$$

Then, for each i , the correspondence $f \mapsto F_i$ defines a homomorphism $\psi_i : \Gamma \rightarrow \mathrm{GL}(W_i/W_{i-1})$. Since $\dim(W_i/W_{i-1}) \leq 2h - 1$, the image of ψ_i is cyclic (trivial if $h \geq 3$) by Theorem 3.

Since the image ψ_i is abelian, we get $\psi_i(f) = I$ for all $f \in [\Gamma, \Gamma]$ and for all i . Hence, for any $f \in [\Gamma, \Gamma]$, $\psi(f)$ is upper triangular with 1 along the diagonal. The subgroup of $\mathrm{GL}(m, \mathbb{C})$ consisting of such matrices is nilpotent, and the group $[\Gamma, \Gamma]$ is perfect by Theorem 3.5. We conclude from this that $\psi([\Gamma, \Gamma])$ is trivial. \square

Corollary 4.9. *Let R be a compact connected oriented surface of genus $g - 1 \geq 3$ and let $\psi : \mathrm{Mod}(R) \rightarrow \mathrm{GL}(2g, \mathbb{C})$ be a homomorphism. If there is a $\mathrm{Mod}(R)$ -invariant subspace W with $3 \leq \dim(W) \leq 2g - 3$, then ψ is trivial.*

5. UNIQUENESS OF THE SYMPLECTIC REPRESENTATION

Let S be a compact oriented surface of genus $g \geq 3$ and let $\phi : \mathrm{Mod}(S) \rightarrow \mathrm{GL}(2g, \mathbb{C})$ be a homomorphism. We prove in this section that either ϕ is trivial or, with respect to a suitable basis of \mathbb{C}^{2g} , the image of ϕ is equal to $\mathrm{Sp}(2g, \mathbb{Z})$ with respect to a suitable basis of \mathbb{C}^{2g} . Theorem 1 will follow from this. Recall that for a simple closed curve a , E_λ^a denotes the eigenspace corresponding to an eigenvalue λ of $L_a = \phi(t_a)$ and $\lambda_\#$ denotes the multiplicity of λ .

Lemma 5.1. *Let $g \geq 4$, let a be a nonseparating simple closed curve on S and let λ be an eigenvalue of L_a . If $\lambda_{\#} \geq 3$ then $\dim(E_{\lambda}^a) \geq 2g - 1$. In particular, $\lambda_{\#} \geq 2g - 1$.*

Proof. Let b be a nonseparating simple closed curve on S intersecting a at one point, and let R denote the complement of a regular neighborhood of $a \cup b$, so that it is a subsurface of genus $g - 1$. Then, $\text{Mod}(R)$ injects into $\text{Mod}(S)$. By identifying $\text{Mod}(R)$ with its image, we assume that $\text{Mod}(R)$ is a subgroup of $\text{Mod}(S)$.

Suppose first that $\dim(E_{\lambda}^a) \leq 2g - 3$. Define a subspace W by

$$W = \begin{cases} \ker(L_a - \lambda I)^{\lambda_{\#}}, & \text{if } \lambda_{\#} \leq 2g - 3, \\ \ker(L_a - \lambda I)^3, & \text{if } \lambda_{\#} \geq 2g - 2 \text{ and } \dim(E_{\lambda}^a) = 1, \\ \ker(L_a - \lambda I)^2, & \text{if } \lambda_{\#} \geq 2g - 2 \text{ and } \dim(E_{\lambda}^a) = 2, \\ E_{\lambda}^a, & \text{if } \lambda_{\#} \geq 2g - 2 \text{ and } 3 \leq \dim(E_{\lambda}^a) \leq 2g - 3. \end{cases}$$

Since elements of $\text{Mod}(R)$ commute with the Dehn twist t_a , the subspace W is $\text{Mod}(R)$ -invariant and its dimension satisfies $3 \leq \dim(W) \leq 2(g - 1) - 1$. Since the genus of R is $g - 1 \geq 3$, $\phi(\text{Mod}(R))$ is trivial by Corollary 4.9. Since t_a is conjugate to some Dehn twist in $\text{Mod}(R)$, we get that $L_a = I$. This says, in particular, that $\dim(E_{\lambda}^a) = 2g$, which is a contradiction.

Suppose now that $\dim(E_{\lambda}^a) = 2g - 2$. If $E_{\lambda}^a \neq E_{\lambda}^b$, then $E_{\lambda}^a \cap E_{\lambda}^b$ is a $\text{Mod}(R)$ -invariant subspace and its dimension is either $2g - 3$ or $2g - 4$. Hence, ϕ is trivial on $\text{Mod}(R)$ by Corollary 4.9. We conclude again that $L_a = I$, obtaining a contradiction. If $E_{\lambda}^a = E_{\lambda}^b$ then by Lemma 4.3 the eigenspace E_{λ}^a is a $\text{Mod}(S)$ -invariant subspace of dimension $2g - 2$. Hence, $0 \subset E_{\lambda}^a \subset \mathbb{C}^{2g}$ is a $\text{Mod}(S)$ -invariant flag. Now Lemma 4.8 applies to conclude that ϕ is trivial, arriving at a contradiction again. \square

Lemma 5.2. *Let $g \geq 4$, let a be a nonseparating simple closed curve on S and let λ be an eigenvalue of L_a . If $\dim(E_{\lambda}^a) = 2g - 1$ then $\lambda = 1$.*

Proof. Let $a = c_1$. Choose six nonseparating simple closed curves $c_2, c_3, c_4, c_5, c_6, c_7$ on S such that we have a lantern relation

$$(8) \quad t_{c_1} t_{c_2} t_{c_3} t_{c_4} = t_{c_5} t_{c_6} t_{c_7}.$$

Hence,

$$(9) \quad L_{c_1} L_{c_2} L_{c_3} L_{c_4} = L_{c_5} L_{c_6} L_{c_7}.$$

The subspace $\bigcap_{i=1}^7 E_{\lambda}^{c_i}$ has positive dimension. Let v be a nonzero vector in this intersection. Evaluating both sides of (9) to v gives $\lambda^4 v = \lambda^3 v$, concluding $\lambda = 1$. \square

5.1. Proof of Theorem 1 for $g \geq 4$. We give the proof in several steps. Let a be a nonseparating simple closed curve on S and let $L_a = \phi(t_a)$.

Step 1: We claim that L_a has only one eigenvalue. First of all, L_a has at most two eigenvalues by Corollary 4.6. Suppose that it has two eigenvalues λ and μ with $\lambda_{\#} \geq \mu_{\#}$. Since $\lambda_{\#} + \mu_{\#} = 2g$, we have $\mu_{\#} \leq g$ and $\lambda_{\#} \geq g$.

Then, we get from Lemma 4.5 that $\mu = 1$, so that $\lambda \neq 1$. On the other hand, by Lemma 5.1, $\lambda_{\#} = 2g - 1$ and the dimension of the eigenspace E_{λ}^a is $2g - 1$. Now Lemma 5.2 implies that $\lambda = 1$, giving the desired contradiction. Therefore, L_a has only one eigenvalue, say λ .

By Lemma 5.1, the dimension of the eigenspace E_{λ}^a is either $2g - 1$ or $2g$.

Step 2: Suppose first that $\dim(E_{\lambda}^a) = 2g$, i.e., $L_a = \lambda I$. For any non-separating simple closed curve x , the Dehn twist t_x is conjugate to t_a , so that L_x is conjugate to L_a , implying that $L_x = \lambda I$. Since the mapping class group $\text{Mod}(S)$ is generated by Dehn twists about nonseparating simple closed curves, it follows that the image of ϕ is cyclic. Since the mapping class group $\text{Mod}(S)$ is perfect, the image of ϕ is trivial.

Step 3: Suppose finally that $\dim(E_{\lambda}^a) = 2g - 1$. Lemma 5.2 implies that $\lambda = 1$. Thus, the Jordan form of L_a is as in (4). Now by Lemma 4.7, with respect to a basis of \mathbb{C}^{2g} , we have $L_{a_i} = A_i$ and $L_{b_i} = B_i$ for each $i = 1, 2, \dots, g$.

Step 4: Let \bar{S} be the surface obtained from S by gluing a disk along each boundary component and let $\pi : \text{Mod}(S) \rightarrow \text{Mod}(\bar{S})$ be the surjective homomorphism obtained by extending diffeomorphisms by the identity over the glued disks. Consider the curves illustrated in Figure 2. Let $e \in \{e_0, e_1, \dots, e_p, f_0, f_1, \dots, f_r\}$. Since e intersects b_1 at one point and is disjoint from all a_i and b_j , $j \geq 2$, the matrix $L_e = \phi(t_e)$ commutes with all A_i and all B_j , and satisfy the braid relation $L_e B_1 L_e = B_1 L_e B_1$. Since L_e is conjugate to L_a , it has only one eigenvalue $\lambda = 1$. From Lemma 2.5, we get that $L_e = A_1$. It now follows from Proposition 3.8 that the kernel of π is contained in the kernel of ϕ . Hence, ϕ induces a homomorphism $\bar{\phi} : \text{Mod}(\bar{S}) \rightarrow \text{GL}(2g, \mathbb{C})$.

Step 5: Note also that $\phi(t_{a'_{g-1}}) = A_{g-1} = \phi(t_{a_{g-1}})$, giving $\pi(t_{a_{g-1}}^{-1} t_{a'_{g-1}}) = I$. A celebrated result of Johnson [11] says that the Torelli subgroup of $\text{Mod}(\bar{S})$ is generated normally by $\pi(t_{a_{g-1}}^{-1} t_{a'_{g-1}})$. Therefore, the Torelli subgroup of $\text{Mod}(\bar{S})$ is contained in the kernel of $\bar{\phi}$, so that $\bar{\phi}$ induces a map $\varphi : \text{Sp}(2g, \mathbb{Z}) \rightarrow \text{GL}(2g, \mathbb{C})$.

$$\begin{array}{ccc}
 \text{Mod}(S) & \xrightarrow{\phi} & \text{GL}(2g, \mathbb{C}) \\
 \downarrow \pi & \searrow \bar{\phi} & \\
 \text{Mod}(\bar{S}) & \xrightarrow{\bar{\phi}} & \text{GL}(2g, \mathbb{C}) \\
 \downarrow & \nearrow \varphi & \\
 \text{Sp}(2g, \mathbb{Z}) & &
 \end{array}$$

Step 6: As the matrix L_a has infinite order, $\text{Im}(\varphi) = \text{Im}(\phi)$ is infinite. One may easily check that $(A_1 B_1 A_2 B_2 \cdots A_g B_g)^3 = -I$ so that the kernel of φ does not contain $-I$. From the solution of the congruence subgroup problem for $\text{Sp}(2g, \mathbb{Z})$ (see [19], Corollar 1.), we conclude that the kernel of φ is trivial, so that φ is injective.

This concludes the proof of Theorem 1.

5.2. Sketch of the proof of Theorem 1 for $g = 3$. The proof of Theorem 1 for $g = 3$ requires more detailed analysis of eigenvalues. We only sketch the proof. Note that in this case $\phi : \text{Mod}(S) \rightarrow \text{GL}(6, \mathbb{C})$.

Let a and b be two nonseparating simple closed curves intersecting at one point, let R be the complement of a regular neighborhood of $a \cup b$, and let c_1 and c_2 be two simple closed curves on R intersecting each other at one point.

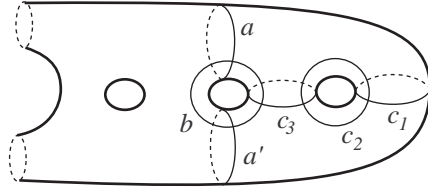


FIGURE 3. Curves on the surface of genus 3.

In this case, L_a has at most two eigenvalues by Corollary 4.6 again. Suppose first that L_a has two eigenvalues λ and μ with $\lambda_{\#} \geq \mu_{\#}$. Then $\lambda_{\#} \geq 4$, $\mu_{\#} \leq 2$, $\mu = 1$ and $\dim(E_{\mu}^a) = \mu_{\#}$. Let $r = \dim(E_{\lambda}^a)$. Consider the following $\text{Mod}(R)$ -invariant flags:

- $0 \subset \ker(L_a - \lambda I)^3 \subset \mathbb{C}^6$ if $r = 1$,
- $0 \subset \ker(L_a - \lambda I) \subset \ker(L_a - \lambda I)^2 \subset \mathbb{C}^6$ if $r = 2$,
- $0 \subset E_{\lambda}^a \subset \mathbb{C}^6$ if $r = 3$.

Therefore, in the cases $r \leq 3$, Lemma 4.8 gives that the commutator subgroup of $\text{Mod}(R)$ is in the kernel of ϕ . In particular, $\phi(t_{c_1} t_{c_2}^{-1}) = I$. Since the normal closure of $t_{c_1} t_{c_2}^{-1}$ in $\text{Mod}(S)$ is $\text{Mod}(S)$ (c.f. Theorem 3.6), the map ϕ is trivial, so that $L_a = I$, a contradiction. In the case $r = 4$, the flag $0 \subset E_{\lambda}^a \cap E_{\lambda}^b \subset E_{\lambda}^a \subset \mathbb{C}^6$ is $\text{Mod}(R)$ -invariant if $E_{\lambda}^a \neq E_{\lambda}^b$, and the flag $0 \subset E_{\lambda}^a \subset \mathbb{C}^6$ is $\text{Mod}(S)$ -invariant if $E_{\lambda}^a = E_{\lambda}^b$, both implying that ϕ is trivial, a contradiction again.

Suppose now that $r = 5$. Fix a basis of \mathbb{C}^6 such that $L_a = \begin{pmatrix} \lambda I_5 & 0 \\ 0 & 1 \end{pmatrix}$. Since the first homology group of $\text{Mod}(S)$ is trivial, $\lambda^5 = 1$. For any simple closed curve x disjoint from a , the matrix of L_x is equal to $L_x = \begin{pmatrix} * & 0 \\ 0 & \gamma(x) \end{pmatrix}$ for some nonzero complex number $\gamma(x)$. If x is a nonseparating curve, then $\gamma(x)$ is either 1 or λ , as it is an eigenvalue of L_x .

Consider the curves illustrated in Figure 3. If $\gamma(c_1) = 1$ then $E_1^{c_1} = E_1^a$. By the braid relation, it can be seen that $\gamma(c_2) = 1$. Hence, $E_1^{c_2} = E_1^a$. It follows that $E_1^x = E_1^a$ for all nonseparating x on S , so that E_1^a is $\text{Mod}(S)$ -invariant. One may deduce from this that ϕ is trivial. In particular, $\lambda = 1$. If $\gamma(c_1) = \lambda$ then braid relations between t_{c_i} imply that $\gamma(c_2) = \gamma(c_3) = \lambda$. Now the relation $(t_{c_1} t_{c_2} t_{c_3})^4 = t_a t_{a'}$ and $\lambda^5 = 1$ imply that $\gamma(a') = \lambda^2$, which

is an eigenvalue of $L_{a'}$. On the other hand, since all eigenvalues of $L_{a'}$ are 1 and λ , we conclude that $\lambda = 1$. By these contradictions, L_a must have only one eigenvalue, say λ .

If $\dim(E_\lambda^a) \leq 4$ then one arrives at a contradiction by deducing that ϕ must be trivial. If $\dim(E_\lambda^a) = 6$ then ϕ is trivial. If $\dim(E_\lambda^a) = 5$ then Lemma 4.4 gives $\lambda = 1$. Now the rest of the proof proceeds as in the case $g \geq 4$. \square

6. NON-FAITHFUL REPRESENTATIONS OF $\text{Mod}(S)$: PROOF OF THEOREM 2

Our aim in this section is to prove Theorem 2. We do this by proving a stronger result, Theorem 6.1, which says that certain terms in the derived series, defined below, of the Torelli subgroup of the mapping class group of a subsurface of genus three are contained in the kernel of ϕ . Since the Torelli subgroups of mapping class groups we consider are not solvable (they contain nonabelian free groups), Theorem 6.1 will certainly imply Theorem 2. Recall that the *Torelli subgroup* of the mapping class group of an orientable surface with at most one boundary component is the subgroup consisting of those mapping classes which act trivially on the first homology of the surface.

For a subsurface X of S with one boundary component, the inclusion $X \hookrightarrow S$ induces a homomorphism $\text{Mod}(X) \rightarrow \text{Mod}(S)$. This map is injective when the genus of X is less than that of S . When this is the case, we identify the group $\text{Mod}(X)$ with its image in $\text{Mod}(S)$ as usual. We denote by T_X (the image of) the Torelli subgroup of $\text{Mod}(X)$.

For a group G , let $G^{(0)} = G$. For each integer $k \geq 1$, we inductively define the k^{th} derived subgroup $G^{(k)}$ of G by

$$G^{(k)} = [G^{(k-1)}, G^{(k-1)}].$$

Recall that a group G is called *solvable* if $G^{(k)} = 1$ for some k .

Theorem 6.1. *Let $n \geq 0$ be an integer. If $g \geq n + 3$ and if $\phi : \text{Mod}(S) \rightarrow \text{GL}(2g + n, \mathbb{C})$ is a homomorphism, then the n^{th} derived subgroup $T_X^{(n)}$ of T_X is contained in the kernel of ϕ for any genus-3 subsurface X of S with one boundary component.*

Proof. We prove the theorem by induction on n . If $n = 0$ then $g \geq 3$, so the result follows from Theorem 1.

Let $n \geq 1$, $g \geq n + 3$ and let $\phi : \text{Mod}(S) \rightarrow \text{GL}(2g + n, \mathbb{C})$ be a homomorphism. Suppose that the theorem holds true for all $k \leq n - 1$. If X and Y are any genus-3 subsurfaces S each with one boundary component, then the classification of surfaces tells us that there is a self-diffeomorphism of S mapping X to Y . It follows that the subgroups $\text{Mod}(X)$ and $\text{Mod}(Y)$ of $\text{Mod}(S)$ are conjugate. It is now easy to conclude that it suffices to prove the theorem for some genus-3 subsurface.

Let a be a nonseparating simple closed curve on S . Since $2g + n \leq 4g - 5$, by Corollary 4.6 the matrix L_a has at most two eigenvalues.

Let b be a nonseparating simple closed curve intersecting a at one point and let R denote the complement of a regular neighborhood of $a \cup b$, so that R is a compact surface of genus $g - 1$. By extending self-diffeomorphisms R to S by the identity, we regard $\text{Mod}(R)$ as a subgroup of $\text{Mod}(S)$.

Case 1: L_a has two eigenvalues. Suppose that L_a has two eigenvalues λ and μ , with $\lambda_{\#} \geq \mu_{\#}$. Recall that $\lambda_{\#}$ denotes the multiplicity of the eigenvalue λ . Since $\lambda_{\#} + \mu_{\#} = 2g + n \leq 3g - 3$, we have $\mu_{\#} < 2g - 3$. By Lemma 4.5, we have $\mu = 1$ and $\dim(E_{\mu}^a) = \mu_{\#}$. The dimension of $\ker(L_a - \lambda I)^{\lambda_{\#}}$ is $\lambda_{\#}$. Let α_1 be a (ordered) basis of $\ker(L_a - \lambda I)^{\lambda_{\#}}$ and let α_2 be a basis of E_1^a . Then $\alpha = \alpha_1 \cup \alpha_2$ is a basis of \mathbb{C}^{2g+n} and in this basis the matrix of L_a is of the form

$$(10) \quad \begin{pmatrix} \star & 0 \\ 0 & I \end{pmatrix},$$

where the size of the identity matrix I is $\mu_{\#}$. We work with this basis.

For any $f \in \text{Mod}(R)$, $\phi(f)$ commutes with L_a so that the subspaces $\ker(L_a - \lambda I)^{\lambda_{\#}}$ and $E_1^a = \ker(L_a - I)$ are $\phi(f)$ -invariant, so

$$\phi(f) = \begin{pmatrix} \star & 0 \\ 0 & F' \end{pmatrix},$$

where F' is of size $\mu_{\#}$. This way we get a homomorphism $\varphi : \text{Mod}(R) \rightarrow \text{GL}(\mu_{\#}, \mathbb{C})$ defined by $\varphi(f) = F'$. Note that since $n \geq 1$, the genus of S satisfies $g \geq 4$, so that the genus of R is $g - 1 \geq 3$. By applying Theorem 3 to φ , we conclude that $F' = I$. It then follows that for any simple closed curve x that is nonseparating on R , the eigenspace of L_x satisfies $E_1^x = E_1^a$. In particular, L_x is of the form (10).

If c and d are two nonseparating simple closed curves on R intersecting at one point, then we have $E_1^c = E_1^d (= E_1^a)$. Now apply Lemma 4.3 to conclude that E_1^a is $\text{Mod}(S)$ -invariant. Since the dimension of E_1^a is less than $2g$, the action of $\text{Mod}(S)$ on E_1^a is trivial. In particular, for any element $f \in \text{Mod}(S)$, the matrix of $\phi(f)$ is

$$\phi(f) = \begin{pmatrix} F & 0 \\ \star & I \end{pmatrix}.$$

It follows that ϕ gives rise to a homomorphism $\bar{\phi} : \text{Mod}(S) \rightarrow \text{GL}(\lambda_{\#}, \mathbb{C})$ defined by

$$\phi(f) = \begin{pmatrix} \bar{\phi}(f) & 0 \\ \star & I \end{pmatrix}.$$

Since $\lambda_{\#} \leq 2g + n - 1$, by the induction hypothesis $T_X^{(n-1)}$ is contained in $\ker(\bar{\phi})$ for some genus-3 subsurface X of S . We conclude from this that $\phi(T_X^{(n-1)})$ is abelian, and hence, $\phi(T_X^{(n)})$ is trivial.

Case 2: L_a has only one eigenvalue. Suppose that L_a has only one eigenvalue, say λ . Let X be any genus-3 subsurface of R with one boundary component. We may consider $\text{Mod}(X)$ as a subgroup of $\text{Mod}(R)$.

We claim that if there is a $\text{Mod}(R)$ -invariant subspace V of dimension r_1 with $3 \leq r_1 \leq 2g + n - 3$, then we are done, namely $\phi(T_X^{(n)})$ is trivial. For the proof of the claim, suppose that we have such a subspace V . Let $r_2 = 2g + n - r_1$ and let α be a (ordered) basis of \mathbb{C}^{2g+n} such that first r_1 elements span V . With respect to this basis, for any $f \in \text{Mod}(R)$,

$$\phi(f) = \begin{pmatrix} \phi_1(f) & \star \\ 0 & \phi_2(f) \end{pmatrix},$$

so that we have two homomorphisms $\phi_1 : \text{Mod}(R) \rightarrow \text{GL}(V) = \text{GL}(r_1, \mathbb{C})$ and $\phi_2 : \text{Mod}(R) \rightarrow \text{GL}(\mathbb{C}^{2g+n}/V) = \text{GL}(r_2, \mathbb{C})$. Since both r_1 and r_2 satisfy $r_1 \leq 2(g-1) + n - 1$ and $r_2 \leq 2(g-1) + n - 1$ and since $g-1 \geq (n-1) + 3$, the derived subgroup $T_X^{(n-1)}$ of T_X is contained in the kernel of both ϕ_i by the induction hypothesis. That is to say,

$$\phi(f) = \begin{pmatrix} I_{r_1} & * \\ 0 & I_{r_2} \end{pmatrix}$$

for any $f \in T_X^{(n-1)}$. It follows that $\phi(T_X^{(n-1)})$ is abelian, and hence $\phi(T_X^{(n)})$ is trivial, proving the claim.

Let us set $r = \dim(E_\lambda^a)$ for the rest of the proof, so that $1 \leq r \leq 2g + n$.

Suppose first that $r \leq 2g + n - 3$. If r is equal to 1 or 2, then $3 \leq \dim(\ker(L_a - \lambda I)^3) \leq 6$. It follows that in this case, there is a $\text{Mod}(R)$ -invariant subspace V with $3 \leq \dim(V) \leq 2g + n - 3$. Hence, by the above claim, $\phi(T_X^{(n)})$ is trivial.

Suppose next that $r = 2g + n$, so that $L_a = \lambda I$. Since t_x is conjugate to t_a for any nonseparating simple closed curve x on S , $L_x = \lambda I$. Since $\text{Mod}(S)$ is generated by such Dehn twists, it follows that the image of ϕ is cyclic. Since the first homology group of $\text{Mod}(S)$ is trivial, we conclude that the image of ϕ is trivial.

Suppose now that $r = 2g + n - 2$. The dimension of E_λ^b is also r . If $E_\lambda^a \neq E_\lambda^b$ then $E_\lambda^a \cap E_\lambda^b$ is a $\text{Mod}(R)$ -invariant subspace of dimension $2g + n - 3$ or $2g + n - 4$. Hence, $T_X^{(n)}$ is contained in the kernel of ϕ by the claim above. If $E_\lambda^a = E_\lambda^b$ then it follows from Lemma 4.3 that E_λ^a is $\text{Mod}(S)$ -invariant. Let β be an ordered basis of \mathbb{C}^{2g+n} whose first r elements form a basis of E_λ^a . For any $f \in \text{Mod}(S)$, the matrix of $\phi(f)$ with respect to β is

$$\phi(f) = \begin{pmatrix} \phi_1(f) & \star \\ 0 & \phi_2(f) \end{pmatrix}.$$

In this way we get two homomorphisms $\phi_1 : \text{Mod}(S) \rightarrow \text{GL}(r, \mathbb{C})$ and $\phi_2 : \text{Mod}(S) \rightarrow \text{GL}(2, \mathbb{C})$. By Theorem 3, the image of ϕ_2 is trivial, and by the induction hypothesis, $T_X^{(n-2)}$ is contained in the kernel of ϕ_1 . Therefore, for

any $f \in T_X^{(n-2)}$,

$$\phi(f) = \begin{pmatrix} I_r & \star \\ 0 & I_2 \end{pmatrix}.$$

It follows that $\phi(T_X^{(n-2)})$ is abelian, and $\phi(T_X^{(n-1)})$ is trivial. In particular, $\phi(T_X^{(n)})$ is trivial.

Suppose finally that $r = 2g + n - 1$. By Lemma 4.4, we get $\lambda = 1$. If $E_1^a = E_1^b$ then by Lemma 4.2 we get that $E_1^x = E_1^a$ for all nonseparating simple closed curves x on S . Since the group $\text{Mod}(S)$ is generated by Dehn twists about nonseparating simple closed curves, it follows that every element of $\text{Mod}(S)$ acts trivially on E_1^a . With respect to a basis of \mathbb{C}^{2g+n} whose first $r = 2g + n - 1$ elements belong to E_1^a , the matrices of L_a and L_b are of the form

$$\begin{pmatrix} I_r & \star \\ 0 & 1 \end{pmatrix}.$$

It follows that $L_a L_b = L_b L_a$. From the braid relation $L_a L_b L_a = L_b L_a L_b$, we get $L_a = L_b$, and so $\phi(t_a t_b^{-1}) = I$. Since the normal closure of $t_a t_b^{-1}$ in $\text{Mod}(S)$ is the whole group, we conclude that ϕ is trivial, which is a contradiction to $\dim(E_1^a) = 2g + n - 1$.

Therefore, we have $E_1^a \neq E_1^b$. Now, by Lemma 4.7, we have $L_{a_i} = \begin{pmatrix} A_i & 0 \\ 0 & I_n \end{pmatrix}$ and $L_{b_i} = \begin{pmatrix} B_i & 0 \\ 0 & I_n \end{pmatrix}$ for a suitable basis of \mathbb{C}^{2g+n} . It can now be concluded as in the proof of Theorem 1 that the kernel of $\text{Mod}(S) \rightarrow \text{Sp}(2g, \mathbb{Z})$ is contained in the kernel of ϕ . In particular, T_X , and hence $T_X^{(n)}$, is contained in the kernel of ϕ .

This completes the proof of the theorem. \square

REFERENCES

- [1] Benson Farb, Dan Margalit, *A primer on mapping class groups*. To be published by Princeton University Press.
- [2] Benson Farb, Alexander Lubotzky, Yair Minsky, *Rank-1 phenomena for mapping class groups*. Duke Math. J. **106** (2001), no. 3, 581–597.
- [3] Stephen J. Bigelow, *Braid groups are linear*. J. Amer. Math. Soc. **14** (2001), 471–486.
- [4] Stephen J. Bigelow, Ryan D. Budney, *The mapping class group of a genus two surface is linear*. Algebr. Geom. Topol. **1** (2001), 699–708.
- [5] Joan S. Birman, *Mapping class groups and their relationship to braid groups*. Comm. Pure Appl. Math., **22** (1969), 213–238.
- [6] Max Dehn, *Papers in Group Theory and Topology*. Translated and Introduced by John Stillwell, Springer (1987).
- [7] John Franks, Michael Handel, *Triviality of some representations of $\text{MCG}(S_g)$ in $\text{GL}(n, \mathbb{C})$, $\text{Diff}(S^2)$ and $\text{Homeo}(\mathbb{T}^2)$* , Preprint. arXiv:1102.4584.
- [8] Louis Funar, *Two questions on mapping class groups*. Proc. Amer. Math. Soc. **139** (2011), no. 1, 375–382.
- [9] Sylvain Gervais, *Presentation and central extensions of mapping class groups*. Trans. Amer. Math. Soc. **348** (1996), no. 8, 3097–3132.

- [10] Nikolai V. Ivanov, *Mapping class groups*. Handbook of geometric topology, 523–633, North-Holland, Amsterdam, 2002.
- [11] Dennis L. Johnson, *Homeomorphisms of a surface which act trivially on homology*. Proc. Amer. Math. Soc. **75** (1979), 119–125.
- [12] Mustafa Korkmaz, *Low-dimensional linear representations of mapping class groups*. arXiv:1104.4816.
- [13] Mustafa Korkmaz, *On the linearity of certain mapping class groups*. Turkish J. Math. **24** (2000), 367–371.
- [14] Mustafa Korkmaz, *Low-dimensional homology groups of mapping class groups: a survey*. Turkish J. Math. **26** (2002), no. 1, 101–114.
- [15] Mustafa Korkmaz, John D. McCarthy, *Surface mapping class groups are ultrahopfian*. Math. Proc. Cambridge Philos. Soc. **129** (2000), no. 1, 35–53.
- [16] Daan Krammer, *The braid group B_4 is linear*. Invent. Math. **142** (2000), no. 3, 451–486.
- [17] Daan Krammer, *Braid groups are linear*. Ann. of Math. (2) **155** (2002), 131–156.
- [18] W.B.Raymond Lickorich, *A representation of orientable combinatorial 3-manifolds*. Ann. of Math. (3) **76** (1962), 531–540.
- [19] Jens Mennicke, *Zur Theorie der Siegelschen Modulgruppe*. Mathematische Annalen **159** (1965) 115–129.

DEPARTMENT OF MATHEMATICS, MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA,
 TURKEY, AND MAX-PLANCK-INSTITUT FÜR MATHEMATIK, BONN, GERMANY
E-mail address: korkmaz@metu.edu.tr